Fractal dimensions of sediments in nature

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On the basis of fractals, the theory of sound attenuation was modified, in which the viscous wavelength was proposed as a scale. By matching the theory to the data of the sound attenuation measured in the sediment in nature, the lower and the upper cutoffs between which the power law proposed by Katz and Thompson is valid were selected, and the fractal dimension of the sediment was obtained. Finally, the fractal dimension of the medium can be estimated in the lower frequency range as well, where the viscous wavelength is greater than the average radius.

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The experimental data published by us [1] revealed that the sound attenuation in a kind of water-saturated riverbed coarse sand in the higher frequency range (where $\beta R \gg 1$) is greater than the theoretical results obtained from the spherical model [2,3], where R is the average radius of the grains,

$$\beta = \frac{2\pi}{\lambda_1}\left[\frac{\pi f \rho_0}{\eta}\right]^{1/2}, \tag{1}$$

and λ_1, ρ_0, and η are the viscous wavelength, the density, the frequency, and the shear viscosity, respectively. In the frequency range from 0.1 to 1 mHz, $\lambda_1 \sim 10^{-3}-10^{-4}$ cm in water. For sediments in nature, the sizes of the grains in them are in the range of $10^{-2}-10^{-4}$, which means that if we observe them in the scale of λ_1, not only are the shapes of the sands in nature spherical ones, but also their surfaces are not smooth, with many structures full of hump and holes so that the fractal dimension of the medium would be different from the one obtained on the basis of the spherical model. On the other hand, according to the cluster fractal, the medium made out of the spherical grains can be referred to as a porous one, which also has its own fractal structure. Mandelbrot proposed comprehensive fractals [4]; after that, many works, for example, Refs. [5] and [6], gave a summary account of further progresses in this area. Recently, Katz and Thompson (KP) investigated the fractal dimensions of porous sandstones by the method of the scanning electronic microscope, and proposed [7]

$$P = \left[\frac{l_1}{l_2}\right]^{3-D}, \tag{2}$$

where P and D are the porosity and fractal dimension of the medium, and l_1 and l_2 are the lower and upper cutoffs between which the power law of the fractals is valid. Afterwards, a comment on the selection of l_1 and l_2 and a reply to it were given by Roberts and KP, respectively [8]. In this paper, taking the viscous wavelength as a scale, we modified the theory of sound attenuation on the basis of the fractals, then matched the experimental data to the theory obtained; finally, the fractal dimension of the medium to be measured was evaluated.

As we already know, the sound attenuation α is proportional to the dissipation power W, i.e.,

$$a \sim W \; . \tag{3}$$

In an impressible homogeneous fluid, the dissipative power [9] is

$$W = -\oint \left\{\rho_0 \vec{V}\left[\frac{V^2}{2}+\frac{p}{\rho_0}\right] - (\vec{V}\cdot\vec{\sigma})\right\} dS - \int \sigma_{ik}\left[\frac{\partial V_i}{\partial x_k}\right] d\tau \; , \tag{4}$$

which consists of surface and bulk integrals, where $\vec{\sigma}$ is the viscous tensor with the components σ_{ik} and $\vec{V}$ and ρ_0 are the velocity vector and the density of the fluid. In a granular medium or in a porous medium, the dissipation power W consists of W_1 in the fluid and W_s in the solid. If the solid is approximately an ideal one, in which the friction can be neglected, its effect only offers some boundary conditions that the fields in the fluid must satisfy. Thus the dissipation power is almost due to the fluid and the boundary of the solid. As pointed out in Refs. [2,3], the sound attenuation in the granular media is due to the scattering and the viscosity. When the sound wavelength λ is much greater than the average radius of the grains (i.e., $2\pi R/\lambda \ll 1$), the former can be neglected, and only the viscous attenuation is a leading one, which means that the sound wave makes the rigid grains oscillating and results in viscous waves in the fluid. The latter will be dissipated in the boundary layer. In consideration of these points, the dissipation power can be denoted by a form similar to Eq. (4), in which the field quantities p and $\vec{V}$ must satisfy the boundary conditions at the surface of the solid. For the granular media, the first term of the intergrand in the surface integral corresponds to the scattering, which does not predominate in the situation just considered [2,3]. Thus the rest terms can be denoted by W_1.

$$W_1 = \oint (\vec{V}\cdot\vec{\sigma}) dS - \int \sigma_{ik}\left[\frac{\partial V_i}{\partial x_k}\right] d\tau \; ,$$

where the integrals are carried out in the fluid and at the boundary surfaces. Since the velocity of the medium is zero at infinity, they can be denoted by the sum of the integrals at all the surfaces of the grains and in the boundary layers of them, i.e.,

$$W_1=\sum_n \left\{ \oint (\vec{V}\cdot\vec{\sigma})dS_n-\tfrac{1}{2}\eta\int \left[\frac{\partial V_i}{\partial x_k}+\frac{\partial V_k}{\partial x_i}\right]^2 d\tau_n \right\} ,$$

where the relationships between the stress and strain were used. Since all the grains are rigid, $\vec{V}$ should be the same in the surface of each grain, so that the surface integral becomes $(\vec{V}\cdot\langle\vec{\sigma}\rangle_S S)_n$, where $\langle\vec{\sigma}\rangle_S$ is the average of the viscous tensor over the surface of the nth grain. If the sizes of these are larger than λ_1, the contribution of the surface integral will be smaller than the bulk one. Thus we have

$$W_1\approx -\tfrac{1}{2}\eta \ll \left[\frac{\partial V_i}{\partial x_k}+\frac{\partial V_k}{\partial x_l}\right]^2 >_\tau >_n \sum_n \tau_n ,$$

where τ_n is the volume of the boundary layer referred to as the nth grain, $\langle\ \rangle_\tau$ denotes the average in the layer of the nth grain, and $\langle\ \rangle_n$ denotes an average over all the grains, respectively. From Eq. (3), one has

$$\alpha \sim -W \sim \sum_n \tau_n ,$$

which means that the sound attenuation is proportional to the volume of the viscous boundary layers of all the grains, or the pore volume of the porous medium. As we already know, this is a fractal one, the dimension of which can be D. In this paper, we use the viscous wavelength λ_1 as a scale, so that

$$\alpha \sim (\lambda_1)^{-D} .$$

If the medium consists of spherical grains, the sound attenuation and the dimension are α_0 and D_0, respectively, then we have

$$\alpha_0 \sim (\lambda_1)^{D_0} .$$

if the average of the radii for both media are the same, one has

$$\frac{\alpha}{\alpha_0} \sim (\lambda_1)^{D_0-D}$$

and

$$\frac{\alpha}{\alpha_0} \approx (\beta R)^{D-D_0} , \qquad (5)$$

approximately, where R is the average radius of the grains, in which the sizes of them obey a distribution. According to Ref. [10], φ approximately obeys a normal distribution where

$$\varphi = -\ln_2(2R) .$$

From Eq. (5), we can see that the sound attenuation in the medium which consists of nonspherical grains equals the one in the medium which consists of spherical grains, multiplied approximately by a factor of $(\beta R)^{D-D_0}$. Figure 1 shows an application of Eq. (5), in which the theoretical attenuation of a spherical modeling was modified by the fractals as given by Eq. (5). Curves 1 and 2 denote the theoretical results of α_0/f and α/f, which are obtained by Ref. [2] and Eq. (5), respectively, and the symbols denote the data of Ref. [1], where the average radius, the standard deviation, and the porosity are 0.017 cm, 0.041 cm, and 0.45, respectively. By means of the optimization method to match the data to the theoretical results obtained from Eq. (5), $D-D_0=0.36$. This fact shows us that if the shapes of the grains in the medium are nonspherical ones, its fractal dimension will be larger than the medium which consists of spherical grains.

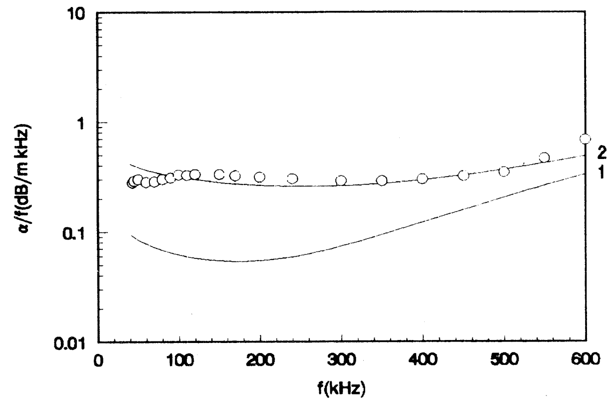

FIG. 1. Sound attenuation α/f (dB/kHz m) vs f (kHz) in riverbed sand. Curve 1: theoretical result on the basis of the spherical-grain model. Curve 2: theoretical result modified by fractals. Porosity: 0.45. Standard deviation: 0.041 cm. Average radius: 0.017 cm. Ratio of densities: 2.62. Symbols: experimental data from Ref. [1].

Now we use Eq. (3) to estimate D_0. From Fig. 1, we regard λ_{11}, which corresponds to $f=550$ kHz, as l_1 and λ_{12}, which corresponds to $f=50$ kHz, as l_2; thus we obtain $D=2.35$ and $D_0=1.99$, which is very close to 2.

In the higher frequency range ($f\geq 550$ kHz) where $\beta R>227$, the experimental data are even larger than the theoretical results obtained by Eq. (5), which may be due to the fact that in the higher frequency range the sound wavelength $\lambda\sim R$. In this situation not only can the scattering not be regarded as a Rayleigh one, but in addition a new fractal scale may occur. In the lower frequency range, where $\lambda_1>R$ and $\lambda\gg R$, we can regard the shapes of all the grains as spherical ones, and can consider the fractal dimension as D_0, which is different from D.

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APPENDIX A

The measurement was carried out in a tank, in which the water-saturated sand to be measured was degassed in order to eliminate the disturbance of air bubbles. A pair of transmitting and receiving transducers were placed in the sand. The distance between them was large enough that the latter was in the far field of the former. A pulse method was used to distinguish the reflection effects from the boundary of the tank. At a series of points r, the sound pressure was measured in a range of frequency, and the following formula was used to obtain the sound attenuation coefficients α, i.e.,

$$p(r_i,f_j)\sim\frac{1}{r_i}e^{\alpha(f_j)r_i} . \tag{A1}$$

where $p(r_i,f_j)$ is the sound pressure measured at the ith point and the jth frequency. Using the data and Eq. (A1), the sound attenuation coefficients $\alpha(f_j)$, $j=1, 2, \ldots, N$ can be obtained, where N is the number of frequencies designated.

APPENDIX B

We denote the experimental data of the sound attenuation coefficient

$$\alpha_e^{(n)}=\alpha_e(f_n) , \tag{B1}$$

and define an objection function

$$F=\sum_n \{\alpha(\tau,R,Q,D-D_0;f_n)-\alpha_e(f_n)\}^2 , \tag{B2}$$

where R, Q, τ, and $D-D_0$ are the average radius, the standard variation, the volume fraction of the grains in sediments, and the fractal-dimensional difference; f_n is the nth frequency, which was designated in the experiment; and $\alpha(\tau,R,Q,D-D_0;f_n)$ is given by Eq. (5) in this paper, into which α_0 can be substituted from Ref. [2]. By means of an optimum computation in order to minimize F, τ, R, Q, and $D-D_0$ can be determined.

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